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ERROR ANALYSIS OF RICHARDSON'S EXTRAPOLATIONS

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Abstract: *We propose estimators of a round off error contained in an approximation for Richardson's extrapolation scheme under finite digit arithmetic. We also propose a stopping criterion, based on consideration of the round off error, for Richardson's extrapolation scheme with respect to risk technical systems (automobile and railway transport, aircrafts, marine and river transport, chemical installations, munitions, information society suffering by terrorism). Usually the error of an approximation is evaluated by a truncation error. However, we can accurately estimate the behavior of this error utilizing both truncation and round off errors under finite digit arithmetic.*

Keywords: *Numerical analysis, error analysis, extrapolation scheme.*

1. INTRODUCTION

The accuracy of the approximation depends on these errors. We know Richardson's extrapolation scheme as a scheme that reduces the accumulated truncation error [3, 6]. When we use this scheme for the actual numerical calculation, it can be done so that this accumulated truncation error may not exist under finite digit arithmetic. This means that the size of the accumulated truncation error becomes smaller than the machine epsilon by Richardson's extrapolation scheme. Therefore, the accuracy of the approximation depends on round off error only.

2. PRELIMINARY CONSIDERATION

We consider to the following initial value problem of ordinary differential equation [1, 2]:

$$y(x_0) = y_0, \frac{dy}{dx} = f(x, y), x \in I, \qquad (1)$$

where I and $y(x)$ are an interval of definition, and the accurate solution, respectively.

The discrete approximation $\eta(x, h_n)$ depends on the step size h_n for $y(x)$.

This h_n can be characterized as follows relation $h_n = I/s_n$, where s_n is an associated sequence for h_n.

We consider that the following Richardson's extrapolation scheme is applied to equation (1):

$$T_k^n = T_{k-1}^n + \frac{T_{k-1}^n - T_{k-1}^{n-1}}{(h_{n-k}/h_n)^\gamma - 1}, \qquad (2)$$

where T_0^n is replaced with $\eta(x, h_n)$. It is well known that T_k^n can be given by the following expansion:

$$T_k^n = y(x) + (-1)^k .a_{k+1}.(h_n.h_{n-1}....h_{n-k})^\gamma + ..., \qquad (3)$$

Therefore, the truncation error τ_k^n included in T_k^n becomes:

$$\tau_k^n = (-1)^k .a_{k+1}.(h_n.h_{n-1}....h_{n-k})^\gamma + ... \qquad (4)$$

We emphasize that the first term of τ_{k-1}^n can be eliminated if T_{k-1}^n and T_{k-1}^{n-1} are substituted for the scheme (2). Next, we use the finite digit arithmetic. It decides to be carried out under finite digit arithmetic when the scheme (2) is applied to the numerical calculation of the practice. Then, we can obtain the value which valid digits length was decided as. Here, the arithmetic of the p floating point l digit is carried out under finite digit arithmetic. We introduce the following symbol to express T_k^n under finite digit arithmetic $\left[T_k^n\right]_l$. Furthermore, the end digit of calculation is expressive of $\varepsilon = p^{-l}$. In general, T_k^n includes an accumulated round off error in finite digits. It can be expressed, when this quantity is shown, with R_k^n. It is shown by equation $R_k^n = \left[T_k^n\right]_l - T_k^n$. Therefore, $\left[R_k^n\right]_l$ shows the round off error that appeared in finite digits. T_k^n in finite digits is shown with $\left[T_k^n + R_k^n\right]_l$. Smaller quantity than ε isn't included by this value.

3. THE EVALUATION OF ROUND OFF ERROR

We consider the accumulated round off error when the scheme (2) is applied. This round off error R_0^n is formed by the calculation of T_0^n, and R_0^n determines R_k^n of T_k^n by the calculation of the recursive scheme (2). We give an estimation of R_k^n of T_k^n. R_k^n becomes the following formula by using the recursive scheme (2). It is shown by equation:

$$R_k^n = R_{k-1}^n + \frac{R_{k-1}^n - R_{k-1}^{n-1}}{\left(h_{n-k}/h_n\right)^\gamma - 1} + l.R_k^n, \qquad (5)$$

where $l.R_k^n$ shows the local round off error which appear by the calculation of the scheme (2).

If $l.R_k^n \ll R_k^n$ than R_k^n can be shown with equation:

$$R_k^n = R_{k-1}^n + \frac{R_{k-1}^n - R_{k-1}^{n-1}}{\left(h_{n-k}/h_n\right)^\gamma - 1}. \qquad (6)$$

Consequently, we can represent R_k^n by:

$$R_k^n = \sum_{i=0}^{k} D_k^{n,i}.R_0^{n-k+i}, \qquad (7)$$

where $D_k^{n,i}$ is defined by the following amount,

$$D_k^{n,i} = \prod_{j=0}^{i-1} \frac{\left(-h_{n-k+j}\right)^\gamma}{\left(h_{n-k+i}\right)^\gamma - \left(h_{n-k+j}\right)^\gamma} \times \qquad (8)$$

$$\times \prod_{j=i+1}^{k} \frac{\left(h_{n-k+j}\right)^\gamma}{\left(h_{n-k+j}\right)^\gamma - \left(h_{n-k+i}\right)^\gamma}.$$

Therefore, $D_k^{n,k}$ becomes:

$$D_k^{n,k} = \prod_{j=0}^{k-1} \frac{\left(-h_{n-k+j}\right)^\gamma}{\left(h_n\right)^\gamma - \left(h_{n-k+j}\right)^\gamma} > 0. \qquad (9)$$

The estimation of R_n^n is:

$$\left|R_n^n\right| \approx \max_{0 \le i \le n} \left|D_n^{n,i}\right|.\left|R_0^i\right|. \qquad (10)$$

In general, if an associated sequence s_n fulfills Toeplitz's condition [2], each $D_k^{n,i}$ becomes as follows:

$$\max_{0 \le i \le n} \left|D_n^{n,i}\right|.\left|R_0^i\right| = D_n^{n,n}.\left|R_0^n\right|. \qquad (11)$$

From equation (7), we suggest the following formula to estimate R_k^n. It is shown be equation:

$$R_k^n = D_n^n.R_0^n. \qquad (12)$$

This suggests as follows. Under finite digit arithmetic, it can't obtain the accuracy of T_k^n beyond the accuracy of $\left[y(x) + R_0^n\right]_l$.

4. A STOPPING CRITERION OF THE EXTRAPOLATION SCHEME

Under finite digit arithmetic, when we look for approximation with the scheme (2), $\left[T_j^n\right]_l$, $j = k, k+1, \ldots$ which has the really same value appears.

We employ this value as an approximation for (1). Eventually, when it becomes equation:

$$\left[T_k^n\right]_l = \left[T_{k+1}^n\right]_l = \ldots = \left[T_n^n\right]_l, \qquad (13)$$

we adopt $\left[T_k^n\right]_l$ as an approximation for the initial value problem (1).

Let consider the stopping criterion proposed. Here, the following criterion is considered for the approximation. It is shown be equation:

$$\left|T_k^n - T_{k-1}^n\right| < \varepsilon . \quad (14)$$

We suppose that T_k^n satisfies the criterion (10). From the scheme (2), we can conduct the following formula:

$$\frac{\left|T_{k-1}^n - T_{k-1}^{n-1}\right|}{\left(h_{n-k}/h_n\right)^\gamma - 1} = \left|a_k\left(h_n h_{n-1}..h_{n-k+1}\right)^\gamma + ..\right| < \varepsilon . \quad (15)$$

Here, if we suppose the following approximate formula:

$$\frac{\left|\tau_{k-1}^n - \tau_{k-1}^{n-1}\right|}{\left(h_{n-k}/h_n\right)^\gamma - 1} \approx \left|a_k\left(h_n h_{n-1}...h_{n-k+1}\right)^\gamma\right|. \quad (16)$$

we can get the next formula:

$$\left|T_k^n - T_{k-1}^n\right| \approx \left|\tau_k^n\right| < \varepsilon . \quad (17)$$

Hence, τ_k^n of T_k^n is shown as follows $\left|\tau_k^n\right| < \varepsilon$. We note that the formula (16) is the assumption that the following assumption is equal to $\left|\tau_k^n\right| << \left|\tau_{k-1}^n\right|$. It is denoted that T_k^n can't be adopted as an approximation when the assumption of the formula (16) isn't satisfied in the criterion (15).

Actually, the problem (1) that the assumption of the formula (16) isn't satisfied in the problem, which has a kind of singularity. We control interval I so that step size can fulfill (16) for such problem [4, 5].

Next, we consider the round off error for the criterion (15) by using the scheme (2), R_k^n and R_{k-1}^n are shown as follows equation:

$$\frac{\left|R_{k-1}^n - R_{k-1}^{n-1}\right|}{\left(h_{n-k}/h_n\right)^\gamma - 1} < \varepsilon . \quad (18)$$

We understand the following fact from the recursive formula (6). It is shown be equation:

$$\frac{\left|R_{k-1}^n - R_{k-1}^{n-1}\right|}{\left(h_{n-k}/h_n\right)^\gamma - 1} \approx \frac{\left|R_{k-1}^n\right|}{\left(h_{n-k}/h_n\right)^\gamma - 1} < \varepsilon . \quad (19)$$

Therefore, we obtain the following conclusion by using the recursive formula (6). If T_k^n is satisfied the criterion (14), $\left[R_k^n\right]_l$ becomes $\left[R_{k-1}^n\right]_l$.

5. OUR STOPPING CRITERION

The stopping criterion that we propose is the model of the criterion (14). We consider it about this criterion (14) as a value under finite digit arithmetic. T_{k-1}^n under finite digit arithmetic can be shown as follows equation:

$$\left[T_{k-1}^n\right]_l = \left[y(x) + \tau_{k-1}^n + R_{k-1}^n\right]_l . \quad (20)$$

From the consideration of the preceding paragraph, when T_{k-1}^n satisfies the criterion (14), τ_{k-1}^n is not present in the calculated digits. Therefore, it becomes equation:

$$\left[T_{k-1}^n\right]_l = \left[y(x) + R_{k-1}^n\right]_l . \quad (21)$$

From the expansion (4), it becomes $\left|\tau_k^n\right| < \left|\tau_{k-1}^n\right|$. Therefore, T_k^n can be shown as the equation:

$$\left[T_k^n\right]_l = \left[y(x) + R_k^n\right]_l . \quad (22)$$

The above shows that if $\left[T_k^n\right]_l$ and $\left[T_{k-1}^n\right]_l$ satisfy the criterion (14), $\left[\tau_k^n\right]_l$ and $\left[\tau_{k-1}^n\right]_l$ aren't included. And, under finite digit arithmetic, $\left[\left|R_{k-1}^n\right| \big/ \left(\left(h_{n-k}/h_n\right)^\gamma - 1\right)\right]_l$ is not included in finite digits. In the formula (2), when T_k^n fulfill the criterion (14), it can indicate $\left[T_k^n\right]_l = \left[T_{k-1}^n\right]_l$.

Therefore, we can get the following three formulas:

$\left[T_k^n\right]_l = \left[y(x) + R_k^n\right]_l$, $\left[T_{k-1}^n\right]_l = \left[y(x) + R_{k-1}^n\right]_l$ and $\left[R_k^n\right]_l = \left[R_{k-1}^n\right]_l$.

We can get the formula (13) from these things. Because, if an extrapolation scheme is applied, $T_j^n\left(j > k\right)$ becomes $\left|\tau_j^n\right| > \left|\tau_{j+1}^n\right|$ and they are not included in finite digits. On the other hand, $T_j^n\left(j > k\right)$ becomes $R_k^n \approx R_j^n$. Further, we consider the criterion (9) for the scheme (2). As for τ_k^n included in T_k^n becomes the formula (16) again. If $\left|a_k\right| < \infty$ is supposed toward the coefficient a_k in the formula (16) and $\left(h_n h_{n-1}...h_{n-k+1}\right)^{\gamma}$ is made fully small, it becomes equation:

$$\frac{\left|T_{k-1}^{n} - T_{k-1}^{n-1}\right|}{\left(h_{n-k}/h_n\right)^{\gamma} - 1} < \varepsilon . \qquad (23)$$

Therefore, it becomes $\left|\tau_{k-1}^{n}\right| \le \varepsilon$ if the formula (23) is fulfilled. This can achieve the above assumptions by making interval I small.

Backward, how I is made small, estimate can do it when coefficients a_j of τ_k^n, are very big value in comparison with the assumptions when the formula (23) doesn't achieve it. In such case, we regard the problem (1) as having numerical singularity.

As for R_k^n included in T_k^n becomes the following formula by the recursive formula (6) and the expansion (7):

$$\frac{\left|R_{k-1}^{n}\right|}{\left(h_{n-k}/h_n\right)^{\gamma} - 1} < \varepsilon . \qquad (24)$$

This denotes that it becomes:

$$\left[R_k^n\right]_l = \left[R_{k-1}^n\right]_l . \qquad (25)$$

Therefore, it was shown that the scheme (2) satisfied the criterion (14). The approximation fulfill the stopping criterion (13) which we proposed, doesn't encompass a truncation error, and it encompasses only a round off error. The accuracy of this approximation can be estimated by the (12).

6. CONCLUSION

The conclusions obtained are summarized as follows:

1. The proposed stopping criterion does not depend on the tolerances, hence, the error contained in the approximation is equal to the estimate of the round of the error risk technical systems.
2. It is possible to find singularity properties of the problem (1), based on the new stopping criterion and controlling interval.
3. As well as the error of approximation is equivalent to accumulated round of error obtained by the new stopping criterion, we confirm a reduction of the error.

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