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Stretch-Contract Operator in the Ellipsoidal Approximation of the Minkowski Sum of Convex Sets

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ABSTRACT The space expansion-contraction operator was originally developed to solve mathematical programming problems. However, it can be successfully applied to solve the problem of ellipsoidal approximation of the information set in the state space analytically specified. In this case, a main property of the operator - space compression is used to minimize the approximating ellipsoid by a multidimensional volume. The paper shows the use of the specified expansion-contraction operator to approximate a set of attainability of the linear control system as an example. The main goal of the paper is to give analytical and geometric representations of the specified operator in order to show its action in the approximation problem. For this purpose, the paper shows an analytical derivation of the operator and a geometric illustration of each parameter of the operator. The results of minimum approximation modeling by this operator compared with other known solutions have been also presented. The simulation results are given both numerically and graphically. Based on the results of comparison, conclusions are made and recommendations are given in the use of ellipsoidal approximation of information sets according to different criteria for minimizing the approximating ellipsoid. Typical examples of ellipsoidal approximation, which show when it is expedient to use the proposed of expansion-contraction operator, have been given.

KEYWORDS stretch-contract operator; state space; attainability set; ellipsoidal approximation; linear control system; multidimensional volume of an ellipsoid; sum of positive degrees of ellipsoid semiaxes; criterion for minimizing an ellipsoid.

I. INTRODUCTION

THE problem of state estimation of the linear controlled system is considered. An object moving in space or some process can be considered as a linear controlled system. For example, a position of the object in space or values of process parameters often called phase coordinates in the state space are considered as a state for the examples selected. Mathematically, such an object or a process is described by linear differential or difference equations. Phase coordinates in the course of the process or functioning of the object change in the desired value under the influence of control and deviate from the desired value within known limits under the impact of uncontrollable factors. It is assumed that the statistical characteristics of these factors are unknown. This is often the case, and in addition, it is required to guarantee existence of state variables in the limited set [1].

A set of possible states of the system can be limited by hyperplanes passing through the previously calculated extreme points – vectors of phase coordinates, the distance between which, taken in pairs, is the biggest. Over time, due to multiple intersections of a set of attainability [1] and a set of measurements, the number of extreme points will become excessive and further processing of measurement information for assessing a state of the system will become impossible. It is possible to build a set of canonical shapes: a hyperparallelepiped, a ball, an ellipsoid. In the first case, we will face an increase in uncertainty of multiple estimate that is a price to pay for simple calculation of parameters of a hyperparallelepiped, approximating states of the system observed. In the second case, calculations of parameters are the simplest, but the estimate can be either better or worse than in the first case. A reasonable compromise is the third case, ellipsoidal approximation. An understanding of this problem is provided by works [1, 2].

An ellipsoid approximating a set of possible states of the observed system can be constructed by different methods, for example, by methods of computational geometry [3] or by analytical methods [4]. For ellipsoidal approximation of convex sets, the apparatus of support functions or of polynomial sublevel sets is often used [5, 6].

The analytical method is considered here. Its advantage is constancy of the algorithm for calculating parameters of an ellipsoid approximating set of possible states of the system. The methods for construction of an ellipsoid by a chosen analytical method may differ in a criterion of minimization. Having chosen a multidimensional volume as such, we obtain a convenient mathematical property for an ellipsoid – invariance of the obtained ellipsoid relative to linear transformations of the operation of set-theoretic summation and intersection of ellipsoids. Methods for obtaining an ellipsoid of minimum volume can be different, but we will show the use of a space stretch-contract operator to obtain a required ellipsoid [7]. Since one of the goals of this paper is to show the use of an stretch-contract operator for visual analysis of the impact of uncontrolled factors on parameters of the approximating ellipsoid, let us consider a one-dimensional case – when only one uncontrollable factor affects the system in a known direction or by a known phase coordinate [8]. The full-dimensional case of a stretch-contract operator is described in [9].

Based on the above material – visual analysis of the impact of an uncontrolled factor on a state of the system – the differences in approximating ellipsoids minimized by different criteria will be shown on the examples.

II. FORMULATION OF THE PROBLEM

Let us follow the work [8]. A linear controlled system for various purposes with the input control action [10-15] and an inexactly known initial state is considered. Let us represent a differential equation describing a motion of the object in the form of a difference equation in the state space. The equation of the system in the phase space of state variables is represented as follows

$$x_{j+1} = Ax_j + Bu_j + w_j, \quad (1)$$

$$x_0 \in E_{st.,0} = \{x_0 : (x_0 - \bar{x}_0)^{\mathrm{T}} \bar{H}_{st.,0}^{-1} (x_0 - \bar{x}_0) \le 1\}, \quad (2)$$

where: $x_j \in X^n$ – a system state vector in n-dimensional Euclidean space; A and B – constants $n \times n$- and $n \times 1$-matrix; $u_j \in \mathbb{R}^1$ – control; $w_j = f_j \zeta_j$ – disturbance action; $f_j \in X^n$ – a unit vector in n-dimensional Euclidean space, the direction of which is taken as constant; $\zeta_j \in \mathbb{R}^1, |\zeta_j| \le d$ – scalar perturbation limited by given constant $d \ge 0$; (A, B) and (A, f_j) pairs are controlled [7]; $j \in T_0$, where $\{T_0 \mid j = 0,1,...,k, \ (k < \infty)\}$ – discrete time; $E_{st.,0} \subset X^n$ – an ellipsoidal set of possible values of the initial state contained in n-dimensional Euclidean space; $\bar{x}_0$ and $\bar{H}_{st.,0}^{\mathrm{T}} = \bar{H}_{st.,0} > 0$ given n-dimensional vector and $n \times n$-matrix respectively. Here *st.* – shortened from *state*.

u_j controls are given in the T_0 whole interval of control forming a program

$$\{u_j \in \mathbb{R}^1, j \in T_0\}. \quad (3)$$

For a set of realizations of perturbation action it is possible to write:

$$\{f_j \zeta_j : |l^{\mathrm{T}} f_j \zeta_j| \le d \sqrt{l^{\mathrm{T}} f_j f_j^{\mathrm{T}} l}, \forall l \in X^n\}. \quad (4)$$

The problem is that by known expressions (1)-(3) to build a set of attainability of this system and approximate it by a minimum ellipsoid under the taken criterion. At this the initial (a priori) estimate $E_{st.,0}$ is known.

Such systems describe the movement of [10-15], for example, an aircraft as a reaction to a change in the position of the steering gear. For example, when turning a rudder in the aircraft, the angular position of the aircraft changes not only in the angle of yaw, but also in the angles of roll and pitch. The dynamics of changes of the phase variables (deviation angles in each of the stated control channels) when changing a position of the rudder in the aircraft of standard composition at the initial position “level flight” will be the most intensive in the course channel, in the roll channel the dynamics will be weaker, and the least intensive reaction will be observed in the pitch channel. The object moving under water behaves in exactly the same way – rudder deflection leads not only to a change in course, but also in roll and pitch with the same dynamics ratio as described above. The dynamic response of the object to control via corresponding channel and connection of the channels is reflected in the matrix A, which describes the dynamics of the system.

When a ship or a car moves on the surface, one of this channel will be away – the pitch will disappear.

We consider, for example, the movement of a car on a highway with curves. The input action is considered to be the turn of the steering wheels. As a result of turning of the steering wheels, the car changes its direction. Depending on the speed of the car, its mass, characteristics of traction of the spots of tire/pavement contact with the road surface, skidding will take place. Skidding is considered as an uncontrolled rotation of the car around a vertical axis passing through a certain point in the car, and a displacement of the car (center of mass) in the lateral direction from the desired trajectory. In this case, the dynamics matrix A reflects characteristics of the movement – the speed and quality of driving tire traction at the assumed mass.

The error in setting of the control action or the uncertainty of any dynamic characteristic reflected in the dynamics matrix A can be considered as the uncertainty of the disturbing action (4).

Thus, the fact that the control action is carried out in one specific channel (in one phase variable of the system) is reflected in the control matrix B – a row corresponding to the desired control channel has nonzero elements, and all other rows are zeros. In the case under consideration (3), the matrix is a column vector, therefore, in this case, it is a matter of only its elements.

III. PROBLEM SOLVING

A set of attainability can be represented in the form of a sum by Minkovsky [16-18] of two sets – ellipsoidal estimate $E_{st.,j}$ of possible initial state of the system (2) and a set of disturbances (4), which is represented a $E_{dst.,j}$ segment in our case, where *dst.* – shortened from *disturbance*.

Their sum will not be an ellipsoid, therefore for further estimation it is necessary to approximate it by ellipsoid $E_{st.,j+1}$. The required explanations are given in Fig. 1.

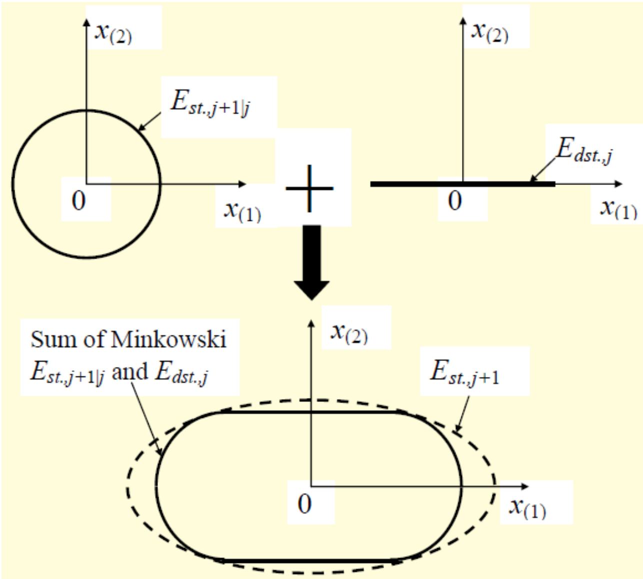

Figure 1. Minkowski summa of a circle $E_{st.,j}$ and a line segment $E_{dst.,j}$ and its ellipsoidal approximation.

Let us perform this approximation with the help of stretch-contract operator, for which we first write down the geometric sum Σ_j of two sets $E_{st.,j}$ and $w_{dst.,j}$ with the help of the apparatus of reference functions:

$$E^*_{j+1} = \{\varepsilon_{j+1} = x_{j+1} - \bar{x}_{j+1} : \; l^T \varepsilon_{j+1} \le \sqrt{l^T H_{j+1|j} l} + \sqrt{l^T f_j f_j^T l}\, d,\ \forall\ l \in X^n\}, \quad (8)$$

where $H_{j+1|j} = A H_{st.,j} A^T$.

Then let us write [9]:

$$H_{st.,j+1} = \gamma_{1,j} H_{j+1|j} + \gamma_{2,j} d^2 f_j f_j^T, \quad \gamma_{1,j}^{-1} + \gamma_{2,j}^{-1} = 1,\ \gamma_{1,j} > 0,\ \gamma_{2,j} > 0, \quad (9)$$

By determination [1], in view of representation (9), we have the following expression for a volume of the ellipsoid:

$$v(E_{st.,j+1}(\gamma_{1,j}, \gamma_{2,j})) = = v(1)(\det H_{st.,j+1}(\gamma_{1,j}, \gamma_{2,j}))^{1/2}, \quad (10)$$

where $v(1)$ – a volume of n-dimensional unit ball in $\mathbb{R}^{n\times n}$, $\det H_{st.,j+1}$ – a matrix determinant $H_{st.,j+1} \in \mathbb{R}^{n\times n}$. With the use of the contraction (expansion) matrix let us represent the matrix (9) in the following form

$$H_{st.,j+1}(\gamma_{1j}, \gamma_{2,j}) = = \gamma_{2,j} \sqrt{H_{j+1|j}}\ R_{\beta_j^{-2}}(\sqrt{H^{-1}_{j+1|j}} f_j) \sqrt{H_{j+1|j}}, \quad (11)$$

where $R_{\beta_j^{-2}}(\sqrt{H^{-1}_{j+1|j}} f_j) = R_j$ – a stretch-contract operator in the direction $\sqrt{H^{-1}_{j+1|j}} f_j$.

$$R_{\beta_j^{-2}}\left(\sqrt{H^{-1}_{j+1|j}} f_j\right) = I - \left(1 - \beta_j^{-2}\right) P_j\left(\sqrt{H^{-1}_{j+1|j}} f_j\right), \quad (12)$$

where $P_j\left(\sqrt{H^{-1}_{j+1|j}} f_j\right) = P_j = \dfrac{\sqrt{H^{-1}_{j+1|j}} f_j f_j^T \sqrt{H^{-1}_{j+1|j}}}{f_j^T H^{-1}_{j+1|j} f_j}$ – projection matrix.

In (12) $I \in \mathbb{R}^{n\times n}$ – a unity matrix and

$$\beta_j^{-2} = 1 + \frac{\gamma_{1,j}}{\gamma_{2,j}} \kappa_j^2, \quad (13)$$

where

$$\kappa_j^2 = f_j^{\mathrm{T}} H_{j+1|j}^{-1} f_j d^2 . \quad (14)$$

In view of the matrix property of the stretch-contract operator

$$\det R_{\beta_j^{-2}}(\sqrt{H_{j+1|j}^{-1}} f_j) = \beta_j^{-2} , \quad (15)$$

for volume (10) let us find

$$v(E_{st.,j+1}(\gamma_{1,j}, \gamma_{2,j})) = v(1)\gamma_{2,j}^{n/2}\beta_j^{-1}\sqrt{\det H_{j+1|j}} . \quad (16)$$

Statement. Let δ_j^+ be a positive equation root

$$n\delta_j^2 + \kappa_j^2((n-1)\delta_j - 1) = 0 , \quad (17)$$

then optimal values $\gamma_{1,j}^*$ and $\gamma_{2,j}^*$ of parameters $\gamma_{1,j}$ and $\gamma_{2,j}$, on which a minimal value of volume (16) of the approximating ellipsoid is achieved, are connected among themselves by ratio:

$$\delta_j^+ = \frac{\gamma_{2,j}^*}{\gamma_{1,j}^*} , \quad (18)$$

that in view of limitation $\gamma_1^{-1} + \gamma_2^{-1} = 1, \gamma_1 > 0, \gamma_2 > 0$ gives

$$\gamma_{1,j}^* = \frac{1+\delta_j^+}{\delta_j^+}, \; \gamma_{2,j}^* = 1+\delta_j^+ . \quad (19)$$

The proof of *statement* follows directly from a necessary condition of function minimum (10) in view of limitation $\gamma_1^{-1} + \gamma_2^{-1} = 1, \gamma_1 > 0, \gamma_2 > 0$.

In new designations let us reformulate expression (9) as follows:

$$H_{j+1} = (1+\delta_j^+)\left(H_{j|j+1} + \frac{d^2}{\delta_j^+} f_j f_j^{\mathrm{T}} \right) . \quad (20)$$

For $n > 1$-dimensional case of perturbation actions, when there is a perturbation action by more than one phase coordinate, it is more convenient to limit perturbation actions not by deviation of the $\zeta_j \in \mathbb{R}^1, |\zeta_j| \le d$ vector module, but an ellipsoid. The matrix of this ellipsoid will be also symmetrical and nonnegative definite, and its rank will be equal to the number of phase coordinates, by which perturbation acts. At any combination of perturbation actions a vector of perturbations is always limited by the named ellipsoid.

To illustrate this case let us reformulate (9) as follows, taking into account that $\gamma_2 = \gamma_1(\gamma_1 - 1)^{-1}$:

$$\begin{aligned} &H_{st.,j+1} = \gamma_{1,j} H_{st.,j} + \gamma_{1,j}(\gamma_{1,j} - 1)^{-1} H_{dst.,j} , \\ &H_{st.,j} = H_{st.,j}^{\mathrm{T}} > 0, H_{dst.,j} = H_{dst.,j}^{\mathrm{T}} \ge 0 \end{aligned} . \quad (21)$$

In (21) it is designated: $H_{dst.,j}$ – ellipsoid matrix limiting a disturbance vector. It should be noted that $H_{dst.,j}$ matrix can be degenerated – its rank is equal to the number of phase coordinates of the system by which the disturbance acts.

In general case the stretch-contract operator is represented as follows [9]:

$$\begin{aligned} &R_{\mathrm{B}_j^{-2}}\left(G_j\right) = I_n + \frac{1}{(\gamma_j - 1)}\sqrt{H_{st.,j}^{-1}} H_{dist.,j} \sqrt{H_{st.,j}^{-1}} = \\ &= I_n - G_j\left(I_m - \mathrm{B}_j^{-2}\right)(G_j^{\mathrm{T}} G_j)^{-1} G_j^{\mathrm{T}} \end{aligned} . \quad (22)$$

$$\mathrm{B}_j^{-2} = I_m + \frac{1}{(\gamma_j - 1)} G_j^{\mathrm{T}} G_j , \quad (23)$$

where $\gamma_j = \gamma_{1,j}$; $G_j = \sqrt{H_{st.,j}^{-1}} S_{dst.,j} \sqrt{\Lambda_{dst.,j}}$; $\Lambda_{dst.,j} \in \mathbb{R}^{m\times m}, m = rankH_{dst.,j}$ – a diagonal matrix of nonzero eigenvalues of $H_{dst.,j}$; $S_{dst.,j} \in \mathbb{R}^{n\times m}$ matrix – a matrix of eigenvectors of the $H_{dist.,j}$ matrix corresponding to nonzero eigenvalues. Finding of the γ optimal value, which supplies an approximating ellipsoid (21) of minimum volume requires a solution of algebraic equation $n+1$-power, in which there is the only positive root, that is an optimal value γ_j [8, 9]:

$$\begin{aligned} &\sum_{i=1}^{n} \frac{1}{\delta_j + \lambda_{(i),j}} = \frac{n}{\delta_j(\delta_j + 1)}; \\ &\lambda_{(i),j} \ne 0_i, 1 \le i \le m; \; \lambda_{(i),j} = 0, \; m < i \le n \end{aligned} . \quad (24)$$

There are no fundamental differences in an action of the stretch-contract operator in one-dimensional and multidimentional cases, therefore let us give the required explanations of its operation to facilitate perception by using a one-dimensional case that is represented in Figs. 2 and 3.

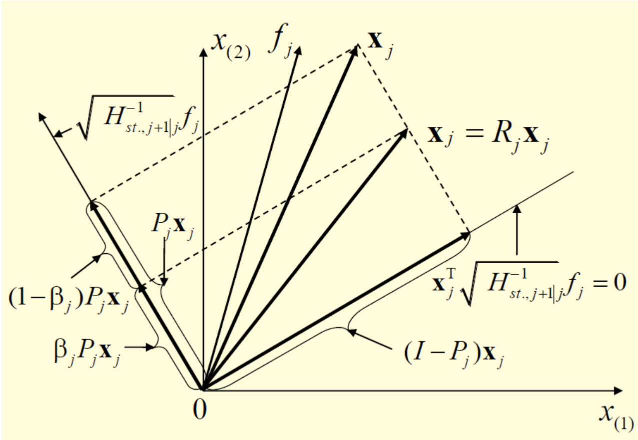

Figure 2. Geometric representation of the space stretch-contract operator

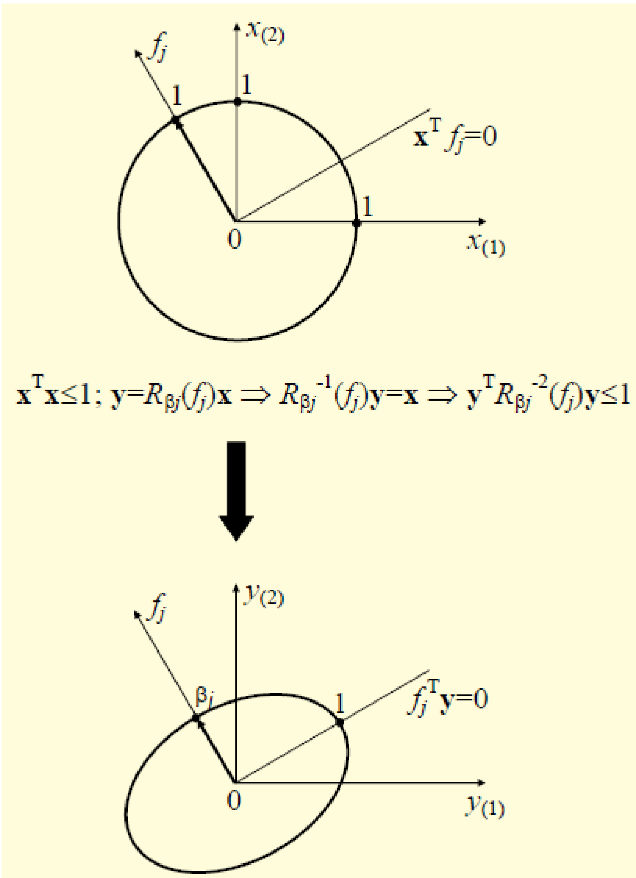

Figure 3. Compress a circle into an ellipse.

In Fig. 2 axes $x_{1,j}$, $x_{2,j}$ are the principal axes of the ellipsoid $E_{j+1|j}$. The vector f_j determines the direction of the disturbing effect. From Fig. 1, it becomes clear that if the vector f_j coincides with the principal axes of the ellipsoid $E_{j+1|j}$, the deformation (stretch-contract) of the approximating ellipsoid $E_{st.,j+1}$ before bringing it to its minimum volume will be carried out by the stretch-contract operator $R\left(\sqrt{H^{-1}_{st.,j+1|j}}f_j\right) = R_j$ in the direction of the axes $x_{1,j}$, $x_{2,j}$, since the vector f_j in this case is an eigenvector of the operator R_j and, as can be seen from (11), (12), the approximating ellipsoid $E_{st.,j+1}$ has the same principal axes as $E_{j+1|j}$. However, due to the arbitrary direction of the disturbance vector f_j relative to the main axes of the ellipsoid $E_{j+1|j}$, the stretch-contract of the approximating ellipsoid for the minimum approximation of the Minkowski sum of the indicated two sets will not be carried out exactly either along the main axes of the ellipsoid $E_{st.,j+1|j}$ or in the direction of the vector f_j. The direction of the stretch-contract in the case of only one disturbing factor is determined from the analytical derivation (11), (12) as $\sqrt{H^{-1}_{st.,j+1|j}}f_j$. From relation (13) it follows that $\beta_j^{-2} > 1$, and further $0 < \beta_j < 1$.

Consider an arbitrary vector $\mathbf{x}_j$ belonging to the state space, in which the family of ellipsoids approximating the sum (8) is located. This vector can be decomposed into two orthogonal vectors: $P_j\mathbf{x}_j$ lying on the axis $\sqrt{H^{-1}_{st.,j+1|j}}f_j$ and a vector $(I-P_j)\mathbf{x}_j$ lying on the axis defined by the expression $\mathbf{x}_j^{\mathrm{T}}\sqrt{H^{-1}_{st.,j+1|j}}f_j = 0$. Under the action of the operator R_j, the projection of the vector $\mathbf{x}_j$ onto the axis $\sqrt{H^{-1}_{st.,j+1|j}}f_j$ will decrease and amount to $\beta_j R_j\mathbf{x}_j$. The projection of the vector $\mathbf{x}_j$ onto the orthogonal axis will not undergo any changes. Fig. 3 shows the deformation of a circle in the direction defined f_j in an ellipsoid. We can assume that there is a mapping of the circle from space X into space Y. In this case, as follows from expressions (9), (11), the ellipsoid is scaled in such a way that the necessary condition $E_{st.,j+1|j} + H_{dst.,j} \subset E_{st.j+1}$ and sufficient condition for the tangency of the sets $E_{st.,j+1|j} + H_{dst.,j}$ and $E_{st.j+1}$ are satisfied at the $2n$ points [9], where is the dimension of the state space. For the multidimensional case, the expressions will be more complex, but their essence will not change.

IV. NUMERICAL SIMULATION

We carry out a numerical simulation of evolution of the approximating ellipsoid in MATLAB and compare the obtained parameters of the ellipsoid – the length of its semiaxes, with those of the ellipsoids optimal according to other criteria, namely, according to the sum of squares of its semiaxes [19], and according to the sum of the fourth powers of its semiaxes [20]. From a mathematical point of view, the main features of ellipsoidal approximation of the sum of two ellipsoids according to these criteria, including a minimum volume criterion, have been considered in [20]. We compare them in view of their use for estimation of uncertainty.

We use formulas (14), (17), and (20) to calculate a minimum volume ellipsoid. To obtain an approximating ellipsoid with a minimum sum of squares of its semiaxes, we use formula [19]:

$$\delta_j^+ = \sqrt{f_j^{\mathrm{T}} f_j d^2 (\mathrm{trace} H_{st.j+1|j})^{-1}} \,. \tag{25}$$

It should be noted that when the disturbing action has not an interval, but an ellipsoidal restriction, that is, it is set by an ellipsoid with the $H_{dst.,j}$ matrix, $traceH_{dist.,j}$ is substituted in the expression (25) for $f_j^{\mathrm{T}} f_j d^2$.

To obtain an approximating ellipsoid with the minimum sum of the fourth powers of its semiaxes, the only positive root of the cubic equation [20] should be found:

$$D_j \delta_j^3 + B_j \delta_j^2 - B_j \delta_j - C_j = 0 \,, \tag{26}$$

where: $C_j = traceH_{dst.,j}^2$, $B_j = trace(H_{st.,j+1|j} H_{dst.,j})$, $D_j = traceH_{st.,j+1|j}^2$.

Obtaining the approximating ellipsoids minimum by the stated criteria – volume, a sum of squares and a sum of fourth powers of the principal semiaxes, differs in computational complexity and requirements for summable ellipsoids. It is also obvious that the resulting approximating ellipsoids will be different. All of these features are considered below.

It is easy to notice that the simplest way from the computational point of view is to obtain an approximating ellipsoid with the minimum sum of squares of the principal semiaxes. With the increase in the dimension of summable ellipsoids, an expression (25) remains unchanged – only the number of summable components under the square root sign increases. In practical use, the expression (25) can be considered as independent of the dimension of summable ellipsoids.

It is more difficult to obtain an approximating ellipsoid with the minimum sum of fourth degrees of the principal semiaxes, as for this purpose it is necessary to find the roots of a cubic equation (26) and to search among them for the unique real positive root. However, computational complexity of solving the equation (26) with the increase in the dimension of summable ellipsoids increases only due to the number of summable components for obtaining coefficients of the cubic equation, and in practical use it can be considered as independent of the state-space dimension.

Finding an approximating minimum volume ellipsoid, as follows from (24), depends on the dimension of summable ellipsoids [9], and in case when both summable ellipsoids have dimension 3 and higher, is the most complex. However, in the important special case (17), when one of the summable ellipsoids is degenerated into a segment, computational complexity of finding an approximating minimum volume ellipsoid takes an intermediate position between finding the minimum ellipsoids by methods (25) and (26), respectively.

One more property of the approximating minimum volume ellipsoid should be noted. If to expose two initial summable ellipsoids to nondegenerate transformation, i.e. to approximate by an ellipsoid minimum in volume, which is subjected to the inverse transformation, the resulting ellipsoid will coincide with the minimum volume ellipsoid for the case of non-transformed summable ellipsoids [1]. For the criteria according to (25) and (26), this does not take place except for the special case when the summable ellipsoids are similar.

As for the restrictions to summable ellipsoids, the general restriction for methods (24), (25), (26) is that none of the summable ellipsoids can be degenerate to a point [1], [11, 20]. Thus, if such a case can occur, a restriction shall be provided in the algorithm.

Only for the method (24) proposed in this paper there is a requirement of nondegeneracy of at least one of the ellipsoids. This follows from the expression (11), in which it is necessary to use an inverse matrix of one of the summable ellipsoids. For methods (25), (26), there is no such a restriction as a trace of the nonnegatively definite matrix with at least one nonzero eigenvalue will be positive, and expressions (25), (26) will make sense.

Thus, for the special case considered in this paper, which is important for practice – approximation of the sum of an ellipsoid and a segment, there are no practical differences in computational complexity as compared to approximation methods (25), (26). It is easy to satisfy the requirement of nondegeneracy for one of the summable ellipsoids either as shown in [9], or by changing a zero eigenvalue of the matrix to a small value exceeding the accepted accuracy of calculations in the algorithm.

To compare the properties of approximating ellipsoids obtained by methods (17), (25), and (26), let us carry out numerical simulation.

Let $H_{st.,0|1} = \mathrm{diag}[1.0, 1.0]$ be a diagonal matrix, $f_j = [1.0; 0.0]^{\mathrm{T}}$, $d = 1.0$. $u_j = 0$, $\forall j \in T_0$. Next, we assume that $A \equiv diag[1; 1]$. Then $H_{st.,2|1} \equiv H_{st.,1}$ and so on.

Having performed calculations for ten steps, we summarize the obtained parameters – lengths of semiaxes of the ellipsoids minimal by their criterion – in tables 1 and 2.

Table 1. Lengths (in arbitrary units) of the semiaxes of the ellipsoids in the direction of the disturbance vector

Step:	Volume minimized ellipsoid	Sum of semiaxes squares minimized ellipsoid	Sum of semiaxes fourth powers minimized ellipsoid
1	2.1213	2.0301	2.0087
2	3.2112	3.0402	3.0101
3	4.2913	4.0452	4.0540
4	5.3665	5.0483	5.0106
5	6.4388	6.0503	6.0107
6	7.5092	7.0518	7.0107
7	8.5781	8.0529	8.0107
8	9.6461	9.0538	9.0107
9	10.7132	10.0544	10.0107
10	11.7797	11.0550	11.0107

Table 2. Lengths (in arbitrary units) of the semiaxes of the ellipsoids in the perpendicular direction to the disturbance vector

Step:	Volume minimized ellipsoid	Sum of semiaxes squares minimized ellipsoid	Sum of semiaxes fourth powers minimized ellipsoid
1	1.2247	1.3066	1.3526
2	1.3869	1.5538	1.6385
3	1.5176	1.7667	1.9848
4	1.6287	1.3066	2.1016
5	1.7262	1.5538	2.2994
6	1.8137	1.7667	2.4817
7	1.8934	1.9566	2.6517
8	1.9669	2.1297	2.8115
9	2.0352	2.2897	2.9628
10	2.0992	2.4392	3.1068

To make the analysis of simulation results convenient, we present the graphs of lengths of the semiaxes for each ellipsoid for all steps.

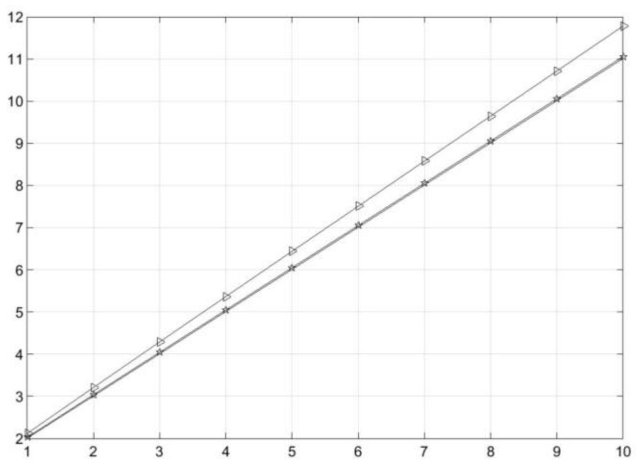

Figure 4. Lengths (in arbitrary units by vertical axis) of the semiaxes of the ellipsoids depending on the step (by horizontal axis) in the direction of the disturbance vector (see table 1).

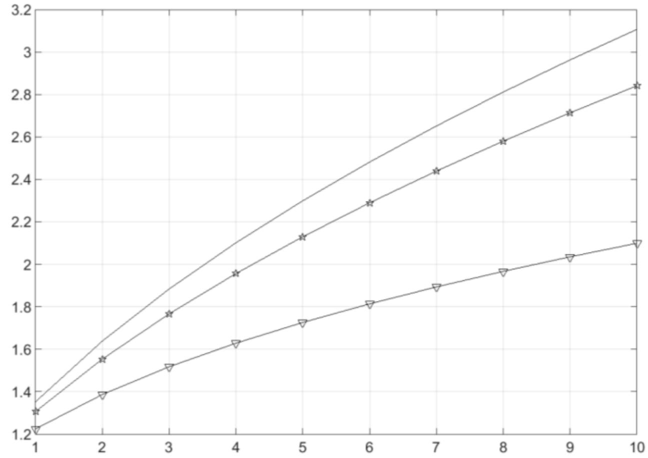

Figure. 5. Lengths (in arbitrary units by vertical axis) of the semiaxes of the ellipsoids depending on the step (by horizontal axis) in the perpendicular direction to the disturbance vector (see table 2).

In Figs. 4 and 5 the triangular marker indicates the graphs of lengths of the semiaxes of the approximating ellipsoid with a minimum volume; the pentagram indicates the graphs of lengths of the semiaxes of the approximating ellipsoid with a minimum sum of squares of its semiaxes; the simple unmarked line indicates the graphs of lengths of the semiaxes of the approximating ellipsoid with the minimum sum of the fourth powers of its semiaxes.

V. DISCUSSION

Now we analyze the results from the tables and graphs. First, we remember the assumed conditions for evolution of the approximating ellipsoid: the initial ellipsoid is a full-dimensional ball (to simplify the simulation, a plane case is taken – a circle); the disturbing action has a constant direction; dynamics of the system is that the ellipsoid is not subject to any transformations from the system itself. Then the $H_{st.,j}$ matrix for any j remains diagonal, which will facilitate the analysis of impact on the approximating ellipsoid of a value and direction of the disturbance action.

It could be seen from Table 1 that the semiaxis of the minimum volume ellipsoid is the biggest in the direction of the disturbance vector in comparison with the semiaxes of ellipsoids optimal according to other criteria. It is analytically obvious and can be shown as follows. It appears for κ_j^2 from the expression (14) that in case of increase in one of the eigenvalues (a square of the semiaxis, in the direction of which the disturbance vector acts) of the $H_{st.,j+1|j}$ matrix the κ_j^2 value decreases. Further considering the equation (17) we can see that with decreasing the κ_j^2 value the δ_j^+ value will also decrease. In its turn, as is seen from expression (20), it will lead to the increased contribution of the disturbing action to one of the eigenvalues of the $H_{st.,j}$ matrix – a square of the semiaxis of the ellipsoid, along which the mentioned disturbing action acts. The approximating ellipsoid will stretch in the direction of the disturbance action, respectively. As a volume of the ellipsoid is directly proportional to a determinant of the matrix that determines it, that is, to the product of its eigenvalues, it becomes obvious that all other semiaxes of the ellipsoid should be as small as possible. More details on properties of a volume of the multidimensional ellipsoid can be found in [16-18].

This is not the case with minimization of the sum of eigenvalues of the matrix of the approximating ellipsoid (squares of its semiaxes) raised to any positive power. In case when the sum of ellipsoids for some reason, for example, due to degeneration of the initial state ellipsoid into a segment and coincidence of directions of this segment and the disturbance vector, also represents a segment, the approximating ellipsoid will be a full-dimensional (nonsingular) ball. However, in case of approximation of the sum of two full-dimensional ellipsoids by Minkowski, the approximating ellipsoid optimal according to a criterion of the minimum sum of powers of its semiaxes will be close to a minimum volume ellipsoid by the lengths of its semiaxes.

VI. ANALYSIS OF SIMULATION RESULTS

The stretch-contract operator contributes a lot to solving problems of approximation of convex sets. For example, if there is an insignificant divergence between the directions of the biggest semiaxis of the initial ellipsoid and the biggest external influence (the biggest dimension of the second convex set in the Minkowski sum), then, as is shown in this paper, the approximating minimum volume ellipsoid will also have the biggest semiaxis in the close direction. This analysis enables to make a decision on the use of an ellipsoid minimum right by volume for approximation of a convex set. In case when the direction of the external disturbance vector is close to orthogonal to the direction of the biggest semiaxis of the initial ellipsoid, then, according to Minkowski, the sum of certain positive power of semiaxes of the approximating ellipsoid can be taken as a criterion for approximation of the sum.

It would seem that in the latter case it is much more advantageous to use minimization of the sum of powers of semiaxes of the approximating ellipsoid as even a degenerated case – the sum of two segments having different directions, can be approximated. There are more complex calculations for minimization of the sum of fourth powers of the approximating ellipsoid as compared to minimization of the volume – finding the unique positive root of a cubic equation – only if $n = 2$ for one set and $n = 1$ for the second set. When dimension of each summable set is $n \geq 2$, in order to obtain the minimum volume ellipsoid, it is necessary to find roots of the algebraic equation of power “four” and larger. However, when it comes to guaranteed estimation, the situation is different. For guaranteed estimation, it is often preferable to have some ‘reassurance’ in the direction of the disturbance action. This is especially pressing in case of increase in the disturbance value without being timely detected by an external observer, or when it comes to human safety. However, the directions in which the disturbance does not act can be estimated without ‘reassurance’. This is the case when the volume of the ellipsoidal estimate is minimized (see table 2 and figure 5). This will enable to avoid reassurance in secondary directions of, for example, object movement, or in the secondary control channels, especially when the system state is simultaneously estimated by different types of canonical convex sets [21, 22].

Degeneracy can be eliminated by adding small values of zero eigenvalues to the ellipsoid matrix.

This paper seems to be much easier for understanding the fact of the matter than the paper [24], although in no way this paper substitutes or even supplements it.

VII. CONCLUSIONS

The above results were obtained by using three sections of analysis: convex, mathematical and matrix, and were further demonstrated by simulation.

The geometric and matrix interpretation of the stretch-contract operator enables with the accuracy sufficient for practice to estimate the uncertainty relations for different parameters or state variables of a certain process or a moving object. Based on this assessment, it is quite easy to select the design parameters of the process under estimation or object to meet the specified requirements. It also gives instructions for the use of approximating ellipsoids obtained according to different criteria.

The limitations on the scope of this paper do not enable to give a sound example of the application of the approximation method proposed in the paper, so let’s restrict ourselves to an overview.

The comparative characteristics of various types of canonical sets used to approximate uncertainty have been given in [23]. There are also practical examples. For instance, the interaction between a robot and a person – when it is necessary to calculate a safe zone in which the robot manipulator can move. Autonomous car movement on the road – there is a great responsibility for the safety of both passengers and other vehicles on the road.

Referring to the handbook [11], it is easy to notice similar problems in calculation of safe maneuvering of ships and aircrafts in the area of busy water or air traffic, respectively. In these cases, the main uncertainty appears along the motion vector, and an ellipsoidal estimate, which considers uncertainty by both main phase coordinate and secondary phase coordinates, is required.

The developed method can also be applied to the nonlinear problems after some minor changes [25, 26].

In the future, the author intends to devote a paper to a specific application of the method proposed herein for calculation of safety zones of moving objects, with the analysis of safety zone dynamics.

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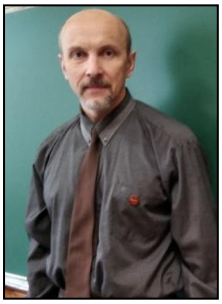

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